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RATIONAL APPROXIMATION METHOD APPROACH TO SECOND ORDER DIFFERENTIAL ALGEBRAIC EQUATIONS

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Abstract —
In this paper, we demonstrate the effectiveness of the rational approximation method when applied to second order differential algebraic equations of time-invariant and time-varying cases. The problems are solved in two methods, one is pade approximation method and another one is rational approximation method which has been developed exclusively to deal with second order systems of differential equations, in particular, time-invariant and time-varying second order differential algebraic equations. The approximate solutions obtained and are compared with the exact solutions of those problems. The errors are presented in graphical form to highlight the efficiency of the rational approximation method.

Keywords — Second order differential algebraic equations of time-invariant and time-varying cases, Singular systems, Pade Approximation, Rational Approximation Method.

INTRODUCTION

Many physical processes are most naturally and easily modelled as singular differential equations. In recent years, there has been an increased interest in several areas in exploiting the advantages of these implicit models. Considerable amount of research has been done on the design of observers for linear time-invariant singular systems (Campbell [24, 25], Dai [46], Hairer et al. [66, 67, 68], Henry Amirtharaj [70], Paul Dhayabaran [5]). However, many singular systems of interest are linear time-invariant or linear time-varying and it is difficult to find the solution of time-varying singular systems.

In science and engineering, singular systems often have to be solved. Although some cases can be solved analytically, the majority of singular systems are too complicated to have analytical solutions. Even when analytical solutions can be found, they are not always useful in practice since the computational cost involved is very high. Here the aim is to demonstrate the accuracy of the rational approximation method, when compared to the pade approximation method.

RATIONAL APPROXIMATION METHOD

The basic principle in designing numerical methods for solving the initial value problem

$$\dot{y} = f(t, y), \quad y(t_0) = y_0 \qquad (1)$$

is that the numerical method must fit the Taylor series expansion of the solution in a given point with the desired accuracy. For a given initial value problem (1), where y is a vector-valued phase variable and t is a scalar (time), the approximate solution obtained in the next time point is denoted as $y_{n+1} \approx y(t_{n+1})$. Denote the time step $h = t_{n+1} - t_n$. Then, consider the Taylor series that approximates the solution:

$$y_{n+1} = \sum_{i=0}^{\infty} \frac{y_n^{(i)}}{i!} h^i. \tag{2}$$

Consider a rational approximation of y_{n+1} as a fraction of two polynomials $P_L(h)$ and $Q_M(h)$:

$$P_L(h) = p_0 + p_1 h + \dots + p_L h^L,$$
$$Q_M(h) = q_0 + q_1 h + \dots + q_M h^M,$$

where L and M are powers of these polynomials. The rational approximation of the solution reads:

$$y_{n+1} = \frac{P_L(h)}{Q_M(h)}. \tag{3}$$

A number of reliable algorithms for finding the rational approximant (3) exist [19]. Here, we use the most straightforward approach. Let M + L = p, where p is a certain natural number. Assuming that (2) and (3) must give the same result up to the error term $O(h^{p+1})$, it follows that

$$\sum_{i=0}^{\infty} \frac{y^{(i)}}{i!} h^i - \frac{P_L(h)}{Q_M(h)} = O(h^{p+1}), \tag{4}$$

which results in a system of equations:

$$\begin{cases} p_0 = y_n q_0, \\ p_1 = h y'_n q_0 + y_n q_1, \\ p_2 = h^2 \frac{y''_n}{2} q_0 + h y'_n q_1 + y_n q_2, \\ \vdots \\ p_L = h^L \frac{y_n^{(L)}}{L!} q_0 + h^{L-1} \frac{y_n^{(L-1)}}{(L-1)!} q_1 + \cdots + y_n q_L, \\ 0 = h^{L+1} \frac{y_n^{(L+1)}}{(L+1)!} q_0 + h^L \frac{y_n^{(L)}}{L!} q_1 + \cdots + h^{L-M+1} \frac{y_n^{(L-M+1)}}{(L-M+1)!} q_M, \\ \vdots \\ 0 = h^{L+M} \frac{y_n^{(L+M)}}{(L+M)!} q_0 + h^{L+M-1} \frac{y_n^{(L+M-1)}}{(L+M-1)!} q_1 + \cdots + h^L \frac{y_n^{(L)}}{L!} q_M. \end{cases} \tag{5}$$

One can notice that this system is complete only if L + M = p, which is exactly the case of the Padé rational approximation. Similarly, the Padé approximant corresponds to the rational approximation of the highest possible order of accuracy. Usually, the notation $R_{L/M}$ or [L/M] is used to denote the powers of polynomials in the Padé approximant with the power of the numerator L and the power of the denominator M. Otherwise, if L + M − p = r, then r > 0, the last r equations from (5) should be omitted, and the r coefficients in P and Q remain free. This option can be used to obtain rational approximants with some desired properties, but their order of accuracy is higher than that with the Padé approximant.

SECOND ORDER DIFFERENTIAL ALGEBRAIC EQUATIONS

In general a second order differential algebraic equations of time-invariant case is represented in the following form

$$K\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t)$$

with initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

where K is an $n \times n$ singular matrix, A and B are $n \times n$ and $n \times p$ constant matrices respectively. $x(t)$ is an n-state vector and $u(t)$ is the p-input control vector and C is an $n \times p$ matrix.

A second order differential algebraic equations of time-varying case is represented in the following form

$$K(t)\ddot{x}(t) = A(t)\dot{x}(t) + B(t)x(t) + C(t)u(t)$$

with initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

where K(t), A(t), B(t), x(t) and u(t) are defined as above, C(t) is an $n \times p$ matrix. The elements (not necessarily all the elements) of the matrices $K(t), A(t)$ and $B(t)$ are time dependent.

NUMERICAL EXAMPLES

In this section, two problems (time-invariant differential algebraic equations and time-varying differential algebraic equations) of second order differential algebraic equations which are taken from real world applications are considered as follows.

Example 4.1

The second order linear time-invariant singular system and B=C=0, becomes $\ddot{x}(t) = A\dot{x}(t)$ with initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \ddot{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \dot{x}$$

with initial conditions $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \dot{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and the exact solution is

$$x_1(t) = 2.0$$
$$x_2(t) = -1 + 2\exp(t/2)$$

Example 4.2

The following second order linear time-varying singular system

$$\begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -t \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t^2 + 2t & 0 \end{bmatrix} \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

is considered with initial conditions $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This problem can be rearranged as

$$\ddot{x}_1 = \dot{x}_1 - 3\dot{x}_2$$
$$\ddot{x}_2 = \dot{x}_2 + 2e^t(t+1)$$

The exact solution is

$$x_1(t) = 1 - t^3 e^t$$
$$x_2(t) = t^2 e^t$$
$$\dot{x}_1(t) = t^2 e^t(-t-3)$$
$$\dot{x}_2(t) = te^t(t+2)$$

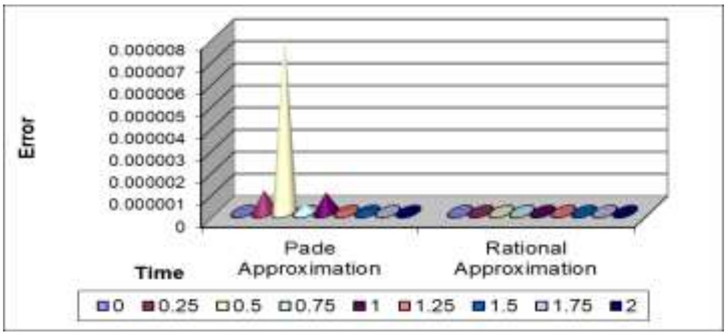

Fig. 1 Error estimation of Example 4.1 at x_2

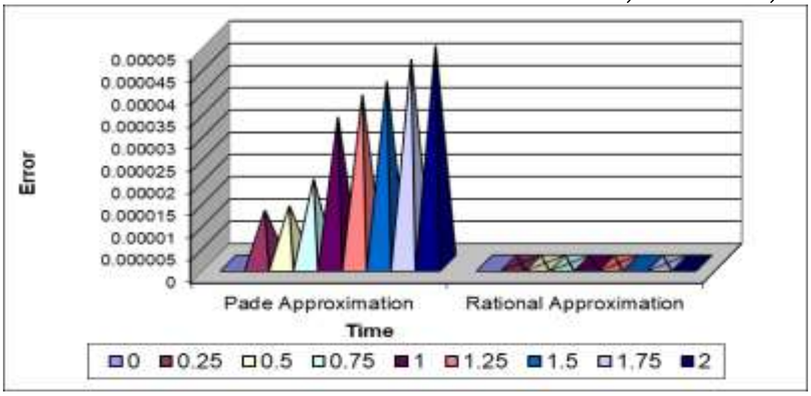

Fig. 2 Error estimation of Example 4.2 at x_1

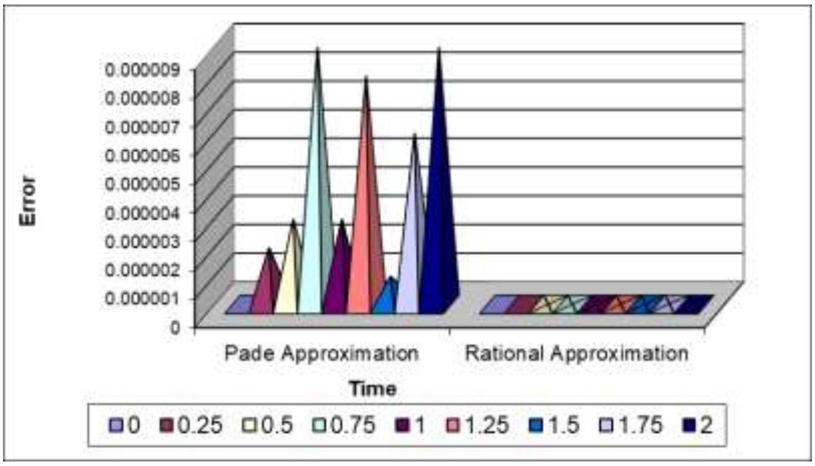

Fig. 3 Error estimation of Example 4.2 at x_2

Using rational approximation method and pade approximation method to solve the problem 4.1 and 4.2, the approximate and exact solutions for x_1 and x_2 are calculated for different values of time 't' and the error between them are shown in figures 1 to 3.

CONCLUSIONS

The Rational approximation method is a powerful, accurate, and flexible tool for solving many types of differential algebraic equations (problems) in scientific computation. The obtained approximate solutions of the second order differential algebraic equations (time-invariant and time-varying) are compared with exact solutions and it reveals that the Rational approximation method works well for finding the approximate solutions. From the figures 1 - 3, one can observe that for most of the time intervals, the absolute error is less (almost no error) in Rational approximation method when compared to the Pade approximation method, which yields a little error, along with the exact solutions.

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